

# Analysis of behavior of a simple eigenvalue of singular system

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**Abstract**—Small perturbations of simple eigenvalues with a change of parameters is a problem of general interest in applied mathematics. The aim of this work is to study the behavior of a simple eigenvalue of singular linear system family

$$\left. \begin{aligned} E(p)\dot{x} &= A(p)x + B(p)u, \\ y &= C(p)x \end{aligned} \right\}$$

smoothly dependent on real parameters  $p = (p_1, \dots, p_n)$ .

**Index Terms**—Singular linear systems, Eigenvalues, Eigenvectors, Perturbation.

## 1. INTRODUCTION

Let us consider a finite-dimensional singular linear time-invariant system

$$\left. \begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \right\} \quad x(t_0) = x_0, \quad (1)$$

where  $x$  is the state vector,  $u$  is the input (or control) vector,  $E, A \in M_n(\mathbb{C})$ ,  $B \in M_{n \times m}(\mathbb{C})$ ,  $C \in M_{p \times n}(\mathbb{C})$  and  $\dot{x} = dx/dt$ . We will represent the systems as quadruples of matrices  $(E, A, B, C)$ . In the case where  $E = I_n$  the systems are standard and we will denote them, as triples  $(A, B, C)$ .

Singular systems are found in engineering systems such as electrical, chemical processing circuit or power systems, aircraft guidance and control, mechanical industrial plants, acoustic noise control, among others, and they have attracted interest in recent years.

Sometimes it is possible to change the value of some eigenvalues introducing proportional and

derivative feedback controls in the system and proportional and derivative output injection. The values of the eigenvalues that can not be modified by any feedback (proportional or derivative) and/or output injection (proportional or derivative), correspond to the eigenvalues of the singular pencil  $\begin{pmatrix} sE - A & B \\ C & 0 \end{pmatrix}$ , that we will simply call eigenvalues of the quadruple  $(E, A, B, C)$ .

Perturbation theory of linear systems has been extensively studied over the last years starting from the works of Rayleigh and Schrodinger [6], and more recently different works as [2], [5],[4], can be found. This treatment of eigenvalues is a tool for efficiently approximating the influence of small perturbations on different properties of the unperturbed system.

Small perturbations of simple eigenvalues with a change of parameters is a problem of general interest in applied mathematics and concretely, this study for the kind of systems under consideration have some interest because in the case where  $m = p < n$ , the most generic types of systems have  $n - m$  simple eigenvalues. The obtained results can be applied to analyze the frequency and damping perturbations in models of flexible structures for example, (see [7], [8]).

In the sequel and without lost of generality, we will consider systems such that matrices  $B$  and  $C$  have full rank and  $m = p < n$ .

## 2. FEEDBACK AND OUTPUT INJECTION EQUIVALENCE RELATION

**Definition 2.1:** Two quadruples  $(E, A, B, C)$  and  $(E', A', B', C')$  are called equivalent if, and only if, there exist matrices  $P, Q \in Gl(n; \mathbb{C})$ ,  $R \in Gl(m; \mathbb{C})$ ,  $S \in Gl(p; \mathbb{C})$ ,  $F_A^B, F_E^B \in M_{m \times n}(\mathbb{C})$  and  $F_A^C, F_E^C \in M_{n \times p}(\mathbb{C})$  such that

$$\begin{aligned} E' &= QEP + F_E^C CP + QBF_E^B, \\ A' &= QAP + F_A^C CP + QBF_A^B, \\ B' &= QBR, \\ C' &= SCP, \end{aligned} \quad (2)$$

or written in a matrix form

$$\begin{pmatrix} E' & B' & 0 & 0 \\ C' & 0 & 0 & 0 \\ 0 & 0 & A' & B' \\ 0 & 0 & C' & 0 \end{pmatrix} = \begin{pmatrix} Q & F_E^C & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & Q & F_A^C \\ 0 & 0 & 0 & S \end{pmatrix} \begin{pmatrix} E & B & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & 0 \end{pmatrix} \begin{pmatrix} P & 0 & 0 & 0 \\ F_E^B & R & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & F_A^B & R \end{pmatrix}.$$

It is easy to check that this relation is an equivalence relation.

A singular system  $(E, A, B, C)$ , for which there exist matrices  $F_E^B$  and/or  $F_E^C$  such that  $E + BF_E^B + F_E^C C$  is invertible is called standardizable, and in this case there exist matrices  $P, Q, F_E^B, F_E^C$  such that  $QEP + QBF_E^B + F_E^C CP = I_n$ . Consequently the equivalent system is standard. Notice that the standardizable character is invariant under the equivalence relation considered.

In the case where the original system is standard and if we want to preserve this condition under the equivalence relation we restrict the operation to the case where  $Q = P^{-1}$ ,  $F_E^B = 0$  and  $F_E^C = 0$ .

**Definition 2.2:** Let  $(E, A, B, C)$  be a system.  $\lambda_0$  is an eigenvalue of this system if and only if

$$\text{rank} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} < \text{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix}.$$

We denote by  $\sigma(E, A, B, C)$  the set of eigenvalues of the quadruple  $(E, A, B, C)$  and we call it the spectrum of the system.

The continuous invariants under this equivalence are the eigenvalues of the system that they are defined as follows.

**Proposition 2.1:** Let  $(E, A, B, C)$  be a system. The spectrum of this system is invariant under equivalence relation considered.

*Proof:* It suffices to observe that

$$\begin{aligned} \text{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} &= \\ \text{rank} \begin{pmatrix} Q & \lambda F_E^C - F_A^C \\ 0 & S \end{pmatrix} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} & \\ \begin{pmatrix} P \\ \lambda F_E^B - F_A^B & R \end{pmatrix}. & \end{aligned}$$

Associated to each eigenvalue there is an eigenvector defined in the following manner:

**Definition 2.3:** i)  $v_0 \in M_{n \times 1}(\mathbb{C})$  is an eigenvector of this system corresponding to the eigenvalue  $\lambda_0$  if and only if, there exist a vector  $w_0 \in M_{m \times 1}(\mathbb{C})$  such that

$$\begin{pmatrix} \lambda_0(E + BF_E^B) - (A + BF_A^B) & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0,$$

for all  $F_E^B, F_A^B$ .

ii)  $u_0 \in M_{1 \times n}(\mathbb{C})$  is a left eigenvector of the system corresponding to the eigenvalue  $\lambda_0$  if and only if, there exist a vector  $\omega_0 \in M_{1 \times p}(\mathbb{C})$  such that

$$\begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \lambda_0(E + F_E^C C) - (A + F_A^C C) & B \\ C & 0 \end{pmatrix} = 0,$$

for all  $F_E^C, F_A^C$ .

**Proposition 2.2:** Let  $\lambda_0$  be an eigenvalue and  $v_0$  an associated eigenvector of the  $(E, A, B, C)$ . Then  $\lambda_0$  is an eigenvalue and  $v_0$  an associated eigenvector of  $(E + BF_E^C + F_E^C C, A + BF_A^B + F_A^C C, B, C)$  for all  $F_E^B, F_E^C, F_A^B, F_A^C$ .

*Proof:* Let  $\bar{w}_0 = w_0 - (\lambda_0 F_E^B - F_A^B)v_0$ .

$$\begin{aligned} \begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \lambda_0 F_E^B - F_A^B & I \end{pmatrix} \begin{pmatrix} v_0 \\ \bar{w}_0 \end{pmatrix} &= \\ \begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ (\lambda_0 F_E^B - F_A^B)v_0 + \bar{w}_0 \end{pmatrix} &= \\ \begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} &= \\ \begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

**Proposition 2.3:** Let  $\lambda_0$  be an eigenvalue and  $u_0$  an associated left eigenvector of the  $(E, A, B, C)$ .

Then  $\lambda_0$  is an eigenvalue and  $u_0$  an associated left eigenvector of  $(E + BF_E^C + F_E^C C, A + BF_A^B + F_A^C C, B, C)$  for all  $F_E^B, F_E^C, F_A^B, F_A^C$ .

*Proof:* Analogous to the proof of proposition 2.2, taking  $\bar{w}_0 = w_0 - u_0(\lambda_0 F_E^C - F_A^C)$ . ■

**Remark 2.1:** Unlike the case of triples of matrices  $(E, A, B)$  (see [4]) if  $\lambda_0$  is an eigenvalue of the quadruple  $(E, A, B, C)$  it is not necessarily a generalized eigenvalue of the pair  $(E, A)$ .

*Example 2.1:* Let  $(E, A, B, C)$  be a system with  $E = I, A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \end{pmatrix}$ .

$$\det \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} = -3\lambda + 3 = 0.$$

Then, the eigenvalue of the system is  $\lambda = 1$ . Observe that  $v_0 = (-3, 3)^t$  is an eigenvector associated to  $\lambda = 1$  (there exist  $w_0 = 1$ ).

But  $\det(\lambda E - A) = \lambda(\lambda - 2)$ , so the eigenvalues of the pair  $(E, A)$  are  $\lambda_1 = 0$  and  $\lambda_2 = 2$ .

**Definition 2.4:** An eigenvalue  $\lambda_0$  of the system  $(E, A, B, C)$  is called simple if and only if verifies the following conditions

$$\begin{aligned} \text{i) } & \text{rank} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} = \\ & \text{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} - 1, \\ & \text{and} \\ \text{ii) } & \text{rank} \begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} = \\ & \text{rank} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} + \text{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix}. \end{aligned}$$

**Proposition 2.4:** The simple character is invariant under equivalence relation considered.

*Proof:* Considering

$$\mathbf{Q} = \begin{pmatrix} Q & \lambda_0 F_E^C - F_A^C & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & F_E^C & Q & \lambda_0 F_E^C - F_A^C \\ 0 & 0 & 0 & S \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} P & 0 & 0 & 0 \\ \lambda_0 F_E^B - F_A^B & R & 0 & 0 \\ 0 & 0 & P & 0 \\ F_E^B & 0 & \lambda_0 F_E^B - F_A^B & r \end{pmatrix}$$

Therefore,

$$\text{rank } \mathbf{Q} \begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} \mathbf{P} =$$

$$\text{rank} \begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix}.$$

**Proposition 2.5:** Let  $\lambda_0$  be a simple eigenvalue of the standard system  $(A, B, C)$ . Then, there exist an associate eigenvector  $v_0$  and an associate left eigenvector  $u_0$  such that  $u_0 v_0 = 1$ .

*Proof:* If  $\lambda_0$  is a simple eigenvalue, the system can be reduced to  $\left( \begin{pmatrix} A_1 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 \end{pmatrix} \right)$ , with rank  $\begin{pmatrix} \lambda_0 I - A_1 & B_1 \\ C_1 & 0 \end{pmatrix} = n - 1$ .

In this reduced form it is easy to observe that  $v_0 = (0, \dots, 0, 1)^t$  is an eigenvector and  $v_0 = (0, \dots, 0, 1)$  is a left eigenvector verifying  $u_0 v_0 = 1$ . Now, taking into account propositions 2.2 and 2.3, we can check easily that  $P v_0$  is an eigenvector of the system  $(A, B, C)$  and  $u_0 P^{-1}$  is a left eigenvector for some invertible matrix  $P$ . ■

**Remark 2.2:** In general, for singular systems this result fails, as we can see in the following example.

*Example 2.2:* Let  $(E, A, B, C)$  be a singular system with  $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$ . It is easy to observe that  $\lambda_0 = 3$  is a simple eigenvalue of this system and all possible eigenvectors are  $v_0 = (\alpha, 0)^t$  with  $\alpha \neq 0$  and all possible left eigenvectors are  $u_0 = (0, \beta)$  with  $\beta \neq 0$ . Clearly  $u_0 v_0 = 0$ .

But, we have the following more general result.

**Proposition 2.6:** Let  $\lambda_0$  be a simple eigenvalue of the singular system  $(E, A, B, C)$  with  $m = p = 1$  and rank  $\begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} = n + 1$ . Then, there exist an associate eigenvector  $v_0$  and an associate left eigenvector  $u_0$  such that  $u_0 E v_0 \neq 0$ .

*Proof:* If  $\lambda_0$  is a simple eigenvalue

$$\begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \\ v_1 \\ w_1 \end{pmatrix} \neq 0$$

for all  $v_1$  and  $w_1$ . So taking  $v_1 = 0$  and  $w_1 = 0$  we have that  $Ev_0 \neq 0$ .

Suppose now that  $u_0 Ev_0 = 0$ , in this case we have that

$$0 \neq \begin{pmatrix} Ev_0 \\ 0 \end{pmatrix} \in \text{Ker} \begin{pmatrix} u_0 & w_0 \end{pmatrix} = \text{Im} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix}. \quad \text{Then,} \quad \begin{pmatrix} Ev_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_0 A - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \text{ for some } (v_1, w_1) \neq (v_0, w_0) \text{ because } Ev_0 \neq 0.$$

So

$$\begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \\ v_1 \\ w_1 \end{pmatrix} = 0$$

ans  $\lambda_0$  can not be simple. Therefore  $u_0 Ev_0 \neq 0$ . ■

### 3. ANALYSIS OF PERTURBATION OF SIMPLE EIGENVALUES

#### A. Standard systems

We begin studying the case of standard systems in order to make more comprehensive the study. So, we consider systems in the form  $\dot{x} = Ax + Bu$ ,  $y = Cx$  with  $A \in M_n(\mathbb{C})$ ,  $B \in M_{n \times m}(\mathbb{C})$  and  $C \in M_{m \times n}(\mathbb{C})$  represented as a triple of matrices  $(A, B, C)$ .

Let  $(A, B, C)$  be a linear system and assume that the matrices  $A, B, C$  smoothly depend on the vector  $p = (p_1, \dots, p_r)$  of real parameters. The function  $(A(p), B(p), C(p))$  is called a multi-parameter family of linear systems. Eigenvalues of linear system functions are continuous functions  $\lambda(p)$  of the vector of parameters. In this section, we are going to study the behavior of a simple eigenvalue of the family of linear systems  $(A(p), B(p), C(p))$ .

Let us consider a point  $p_0$  in the parameter space and assume that  $\lambda(p_0) = \lambda_0$  is a simple eigenvalue of  $(A(p_0), B(p_0), C(p_0)) = (A_0, B_0, C_0)$ , and

$v(p_0) = v_0$  is an eigenvector, i.e. there exists  $w_0 \in M_{m \times 1}(\mathbb{C})$  such that

$$\left. \begin{aligned} A_0 v_0 - B_0 w_0 &= \lambda_0 v_0 \\ C_0 v_0 &= 0 \end{aligned} \right\}.$$

Equivalently

$$\left. \begin{aligned} (A_0 + B_0 F_A^B) v_0 - B_0 w_0 &= \lambda_0 v_0 \\ C_0 v_0 &= 0 \end{aligned} \right\},$$

$\forall F_A^B \in M_{m \times n}(\mathbb{C})$ .

Now, we are going to review the behavior of a simple eigenvalue  $\lambda(p)$  of the family of standard linear systems.

The eigenvector  $v(p)$  corresponding to the simple eigenvalue  $\lambda(p)$  determines a one-dimensional null-subspace of the matrix operator  $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$  smoothly dependent on  $p$ . Hence, the eigenvector  $v(p)$  (and corresponding  $w(p)$ ) can be chosen as a smooth function of the parameters. We will try to obtain an approximation by means of their derivatives.

We write the eigenvalue problem as

$$\left. \begin{aligned} A(p)v(p) - B(p)w(p) &= \lambda(p)v(p) \\ C(p)v(p) &= 0. \end{aligned} \right\}. \quad (3)$$

Taking the derivatives with respect to  $p_i$ , we have

$$\left. \begin{aligned} \left( \frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} \right) v(p) + \frac{\partial B(p)}{\partial p_i} w(p) &= \\ (A(p) - \lambda(p)I) \frac{\partial v(p)}{\partial p_i} - B(p) \frac{\partial w(p)}{\partial p_i} &= \\ \frac{\partial C(p)}{\partial p_i} v(p) = -C(p) \frac{\partial v(p)}{\partial p_i} & \end{aligned} \right\}.$$

At the point  $p_0$ , we obtain

$$\left. \begin{aligned} \left( \frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} \right) \Big|_{p_0} v_0 + \frac{\partial B(p)}{\partial p_i} \Big|_{p_0} w_0 &= \\ (A_0 - \lambda_0 I) \frac{\partial v(p)}{\partial p_i} \Big|_{p_0} - B_0 \frac{\partial w(p)}{\partial p_i} \Big|_{p_0} &= \\ \frac{\partial C(p)}{\partial p_i} \Big|_{p_0} v_0 = -C_0 \frac{\partial v(p)}{\partial p_i} \Big|_{p_0} & \end{aligned} \right\}. \quad (4)$$

This is a linear equation system for the unknowns

$$\frac{\partial \lambda(p)}{\partial p_i}, \quad \frac{\partial v(p)}{\partial p_i} \quad \text{and} \quad \frac{\partial w(p)}{\partial p_i}.$$

**Lemma 3.1:** Let  $v_0$  and  $u_0$  be an eigenvector and a left eigenvector respectively, corresponding to the

simple eigenvalue  $\lambda_0$  of the system  $(E, A, B, C)$ . Then, the matrix

$$T = \begin{pmatrix} \lambda_0 I - A_0 & B_0 \\ C_0 & 0 \end{pmatrix} + \begin{pmatrix} v_0 u_0 & 0 \\ 0 & 0 \end{pmatrix}$$

has full rank.

*Proof:* It suffices to consider the system in the reduced form  $\left( \begin{pmatrix} A_1 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 \end{pmatrix} \right)$ . ■

**Proposition 3.1:** The system (4) has a solution if and only if

$$\begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0 \quad (5)$$

where  $u_0$  is a left eigenvector for the simple eigenvalue  $\lambda_0$  of the system  $(A_0, B_0, C_0)$ .

*Proof:* The system (4) can be rewritten as

$$\begin{pmatrix} \frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} A_0 - \lambda_0 I & -B_0 \\ -C_0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v(p)}{\partial p_i} \\ \frac{\partial w(p)}{\partial p_i} \end{pmatrix} \Big|_{p_0} \quad (6)$$

We have that (4) has a solution if and only if (6) has a solution.

Premultiplying both sides of the equation (6), by  $(u_0, \omega_0)$

$$\begin{aligned} & \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda(p)}{\partial p_i} I - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \\ & \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda(p)}{\partial p_i} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} - \\ & \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial A(p)}{\partial p_i} & -\frac{\partial B(p)}{\partial p_i} \\ -\frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0. \end{aligned}$$

We obtain a solution for  $\frac{\partial \lambda(p)}{\partial p_i} \Big|_{(\lambda_0; p_0)}$ .

$$\frac{\partial \lambda(p)}{\partial p_i} \Big|_{p_0} = \frac{\begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial A(p)}{\partial p_i} & -\frac{\partial B(p)}{\partial p_i} \\ -\frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}}{u_0 v_0}.$$

Using the normalization condition, that is to say, taking  $v_0$  such that  $u_0 v_0 = 1$ , we have:

$$\frac{\partial \lambda(p)}{\partial p_i} \Big|_{p_0} = \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial A(p)}{\partial p_i} & -\frac{\partial B(p)}{\partial p_i} \\ -\frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}.$$

Knowing  $\frac{\partial \lambda(p)}{\partial p_i} \Big|_{p_0}$  we can deduce  $\frac{\partial v(p)}{\partial p_i} \Big|_{p_0}$ .

First of all, we observe that if  $u_0 v_0 = 1$ , then  $u_0 v(p) \neq 0$  and we can take  $v(p)$  such that  $u_0 v(p) = 1$  (normalization condition, it suffices to take as  $v(p)$  the vector  $\frac{1}{u_0 v(p)} v(p)$ ). So

$$\frac{\partial u_0 v(p)}{\partial p_i} = u_0 \frac{\partial v(p)}{\partial p_i} = 0.$$

Consequently we can consider the compatible equivalent system:

$$\begin{pmatrix} \frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} A_0 - \lambda_0 I + v_0 u_0 & -B_0 \\ -C_0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v(p)}{\partial p_i} \\ \frac{\partial w(p)}{\partial p_i} \end{pmatrix} \Big|_{p_0} \quad (7)$$

In our particular case where  $m = p$ , the system has a unique solution

$$\begin{pmatrix} \frac{\partial v(p)}{\partial p_i} \\ \frac{\partial w(p)}{\partial p_i} \end{pmatrix} \Big|_{p_0} = T^{-1} \begin{pmatrix} \frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}.$$

■

Taking the partial derivative  $\partial^2 / \partial p_i \partial p_j$  on both sides of both equations in the eigenvalue problem (3), we can obtain a second order approximation for eigenvalues.

### B. Singular systems

Now we consider singular systems as in (1) written as quadruple of matrices  $(E, A, B, C)$ .

In this case the eigenvalue problem is written as

$$\left. \begin{aligned} A(p)v(p) - B(p)w(p) &= \lambda(p)E(p)v(p) \\ C(p)v(p) &= 0. \end{aligned} \right\} \quad (8)$$

Taking the derivatives with respect to  $p_i$ , we have

$$\left. \begin{aligned} \left( \frac{\partial \lambda(p)}{\partial p_i} E(p) + \lambda(p) \frac{\partial E(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} \right) v(p) + \frac{\partial B(p)}{\partial p_i} w(p) \\ = (A(p) - \lambda(p)E(p)) \frac{\partial v(p)}{\partial p_i} - B(p) \frac{\partial w(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} v(p) = -C(p) \frac{\partial v(p)}{\partial p_i} \end{aligned} \right\}.$$

At the point  $p_0$ , we obtain

$$\left. \begin{aligned} & \left( \frac{\partial \lambda(p)}{\partial p_i} E_0 + \lambda_0 \frac{\partial E(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} \right) v_0 + \frac{\partial B(p)}{\partial p_i} w_0 \\ & = (A_0 - \lambda_0 E) \frac{\partial v(p)}{\partial p_i} \Big|_{p_0} - B_0 \frac{\partial w(p)}{\partial p_i} \Big|_{p_0} \\ & \frac{\partial C(p)}{\partial p_i} \Big|_{p_0} v_0 = -C_0 \frac{\partial v(p)}{\partial p_i} \Big|_{p_0} \end{aligned} \right\} \quad (9)$$

This is a linear equation system for the unknowns  $\frac{\partial \lambda(p)}{\partial p_i}$ ,  $\frac{\partial v(p)}{\partial p_i}$  and  $\frac{\partial w(p)}{\partial p_i}$ .

Suppose now, systems  $(E, A, B, C)$  with  $m = p = 1$  and  $\text{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} = n + 1$ .

**Lemma 3.2:** Let  $v_0$  and  $u_0$  be an eigenvector and a left eigenvector respectively, corresponding to the simple eigenvalue  $\lambda_0$  of the system  $(E, A, B, C)$ . Then, the matrix

$$T = \begin{pmatrix} \lambda_0 E - A_0 & B_0 \\ C_0 & 0 \end{pmatrix} + \begin{pmatrix} E_0 v_0 u_0 E_0 & 0 \\ 0 & 0 \end{pmatrix}$$

has full rank.

*Proof:* First of all we proof that  $E_0 v_0 u_0 E_0 \neq 0$ .

$$\begin{aligned} & (u_0 \quad \omega_0) \begin{pmatrix} \lambda_0 E_0 - A_0 + E_0 v_0 u_0 E_0 & B_0 \\ C_0 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \\ & = (u_0 \quad \omega_0) \begin{pmatrix} E_0 v_0 u_0 E_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \\ & (u_0 E_0 v_0)^2 \neq 0, \end{aligned}$$

so,  $E_0 v_0 u_0 E_0 \neq 0$ .

In the other hand  $v_0 \notin \text{Ker } E_0 v_0 u_0 E_0$ , because  $0 \neq (u_0 E_0 v_0)^2 = u_0 (E_0 v_0 u_0 E_0 v_0)$ .

Suppose now, that

$$\begin{pmatrix} \lambda_0 E_0 - A_0 + E_0 v_0 u_0 E_0 & B_0 \\ C_0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0$$

for some vectors  $v$  and  $w$ .

Then

$$\begin{aligned} 0 & = (u_0 \quad \omega_0) \begin{pmatrix} \lambda_0 E_0 - A_0 + E_0 v_0 u_0 E_0 & B_0 \\ C_0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \\ & (u_0 \quad \omega_0) \begin{pmatrix} E_0 v_0 u_0 E_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = u_0 E_0 v_0 u_0 E_0 v. \end{aligned}$$

Taking into account that  $u_0 E_0 v_0 \neq 0$  we have that  $u_0 E_0 v = 0$ , so  $E_0 v_0 u_0 E_0 v = 0$ , then  $v$  is an eigenvector of the system corresponding to the eigenvalue  $\lambda_0$  linearly independent of  $v_0$ , but  $\lambda_0$  is simple. ■

**Proposition 3.2:** The system (9) has a solution if and only if

$$(u_0 \quad \omega_0) \begin{pmatrix} \frac{\partial \lambda(p)}{\partial p_i} E_0 + \lambda_0 \frac{\partial E(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \Big|_{p_0} = 0 \quad (10)$$

where  $u_0$  is a left eigenvector for the simple eigenvalue  $\lambda_0$  of the system  $(E_0, A_0, B_0, C_0)$ .

*Proof:* Analogously to the proof of proposition 3.1 we observe that proposition 2.6 permits to clear the unknown  $\frac{\partial \lambda(p)}{\partial p_i}$  from equation (10).

On the other hand, taking into account that  $u_0 E_0 v_0 \neq 0$ , we have that  $u_0 E(p) v(p) \neq 0$  in a neighborhood of the origin. So,  $u_0 E_0 \frac{\partial v(p)}{\partial p_i} = 0$ . Lemma 3.2 permits to obtain  $\frac{\partial v(p)}{\partial p_i}$  and  $\frac{\partial w(p)}{\partial p_i}$ . ■

**Example 3.1:** Consider now, the following two-parameter differentiable family of systems  $(E(p), A(p), B(p), C(p))$  with

$$E(p) = I, \quad A(p) = \begin{pmatrix} p_1 & 0 \\ 1 & 2 + p_2 \end{pmatrix},$$

$$B(p) = \begin{pmatrix} 3 + p_1 + p_2 \\ p_1 \end{pmatrix}, \quad C(p) = (1 + p_1, 1 + p_2).$$

At  $p_0 = (0, 0)$  we have that  $\lambda_0 = 1$  is a simple eigenvalue,  $v_0 = \begin{pmatrix} -3 & 3 \end{pmatrix}^t$  a right eigenvalue (with  $w_0 = (1)$ ) and  $u_0 = (0, 1)$  (with  $(\omega_0 = (1))$ ) a left eigenvector. Then

$$\frac{\partial \lambda}{\partial p_1} = \frac{(0 \quad 1 \quad 1) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}}{(0 \quad 1) \begin{pmatrix} -3 & 3 \end{pmatrix}^t} = -\frac{2}{3},$$

$$\frac{\partial \lambda}{\partial p_2} = \frac{(0 \quad 1 \quad 1) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}}{(0 \quad 1) \begin{pmatrix} -3 & 3 \end{pmatrix}^t} = 3.$$

#### 4. CONCLUSION

In this work families of singular systems in the form  $E(p)\dot{x} = A(p)x + B(p)u$ ,  $y = C(p)x$  smoothly dependent on a vector of real parameters  $p = (p_1, \dots, p_n)$  have been considered. A study of the behavior of a simple eigenvalue of this family of singular linear system is analyzed and a description of a first approximation of the eigenvalues and corresponding eigenvectors have been obtained.

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